

Two Methods for Reducing the Number of Multiplications in Complex Fast Fourier Transforms

D. E. Wallis

Communications Systems Research Section

Computational savings in hardware and software mechanizations of the Fast Fourier Transform (FFT) can be obtained by two methods: the first method, generalizable for N a power of 2, exploits the intrinsic simplicity of multiplication by j (unit imaginary), in addition to the periodicity and half-period negation identities usually employed. The second method, outlined for the case $N = 16$ only, exploits the quadrantal symmetries of the real cosine and sine functions in an implementation of the complex FFT which uses only real multiplications. The first method requires $N/2 \log_2 N/8 + 2$ nontrivial complex multiplications, or 10 complex multiplications at $N = 16$. The second method requires only 12 real-coefficient multiplications at $N = 16$ to achieve the same result, but a generalization to higher N is not presently known.

I. Introduction

The conventional Fast Fourier Transform (FFT), of Refs. 1 and 2, is a highly efficient means of accomplishing the discrete Fourier transform. The current versions of the FFT for N (number of tabular points in the discrete function to be transformed) equal an integer power of 2 are based on matrix factorization and exploitation of the two complex-exponential identities

(periodicity) $e^{-j 2\pi/N nm} \equiv e^{-j 2\pi/N (nm \bmod N)}$ N, m, n integers (1)

(half-period negation) $e^{-j 2\pi/N (n + N/2)} \equiv -e^{-j 2\pi/N n}$ N even (2)

to which might be added (as done in this report) the identity

(identification of $-j$) $e^{-j 2\pi/N (N/4)} \equiv -j$ (3)

The result is often expressed as a signal flowgraph in which the number of complex additions (subtractions)

and complex multiplications is seen to be enormously reduced relative to the number of such operations which would be required in executing the transform verbatim from its usual $N \times N$ -matrix-times- N -vector definition.

The objective of this report is to present two methods for reducing still further the number of multiplications required to implement the FFT. The derivation of the first method (FFT) exploits (3) to obtain a favorable grouping of the multiplications by j . The derivation of the second method—real FFT (RFFT)—exploits four additional identities:

(frequency aliasing)
$$e^{-j 2\pi/N mn} \equiv e^{+j 2\pi/N m (N-n)} \quad (4)$$

(definition of complex exponential)
$$e^{-j 2\pi/N mn} \equiv \left(\cos \frac{2\pi}{N} mn \right) - j \left(\sin \frac{2\pi}{N} mn \right) \quad (5)$$

(cosine in terms of sine)
$$\cos \left[\frac{2\pi}{N} n \right] \equiv \sin \left[\frac{2\pi}{N} \left(n + \frac{N}{4} \right) \right] \quad (6)$$

(sine of full period in terms of quarter period, with symmetry)
$$\begin{cases} \sin \left[90^\circ + \frac{2\pi}{N} n \right] \equiv \sin \left[90^\circ - \frac{2\pi}{N} n \right] \\ \sin \left[-\frac{2\pi}{N} n \right] \equiv -\sin \left[\frac{2\pi}{N} n \right] \end{cases} \quad (7)$$

The derivations for both methods follow parallel developments based on matrix partitioning. The resulting transforms are expressed as flowgraphs for comparison of the methods. At $N = 16$, the RFFT requires 12 real-coefficient multiplications, 72 additions, and 7 *trivial* multiplications by j (or $-j$), the unit imaginary, to accomplish the transform, as opposed to the 10 complex-coefficient multiplications, 64 additions, and 7 multiplications by the unit imaginary required by the FFT.

It is not presently known whether the savings in multiplications of the RFFT over the FFT for $N = 16$ can be obtained in general. The complex FFT used in the comparison is possibly a new form in that its flowgraph has exactly $(N/2) \log_2 (N/8) + 2$ nontrivial multiplications (with $N \log_2 N$ additions) as compared with $(N/2) \log_2 (N/2)$ multiplications usually quoted in the literature. The savings thereby obtained are, however, implicit (if not easily recognized) in other standard forms of the FFT.

II. Discussion of FFT and RFFT Flowgraphs

Flowgraphs for the FFT and RFFT are given for the case $N = 16$ in Figs. 1 and 2, respectively. In both flowgraphs, the complex-valued data vector to be transformed is $(x_0, x_1, \dots, x_{15})$, and the complex-valued transform of the data is the vector $(s_0, s_1, \dots, s_{15})$. Nodes of the graphs having two input signal lines are *summing nodes*: Any signal brought to a summing node via a solid line is added, and any signal brought on a dashed line is subtracted. The signal resulting from the summation (or differencing) operation then flows to subsequent nodes. An integer n appearing inscribed in a node of the FFT (Fig. 1) implies that the result of summation is to be multiplied by the *complex* scalar coefficient $\exp [-j(2\pi/N)n]$ which is, for $N = 16$, $\exp [-j(2\pi/16)n]$. Somewhat correspondingly, an integer n appearing inscribed in a node of the RFFT (Fig. 2) implies multiplication by the *real* scalar coefficient $\sin [(2\pi/N)n]$, which is $\sin [(2\pi/16)n]$ when $N = 16$. It was not possible, in drawing the RFFT diagram, to associate all of the required multiplications with specific nodes. To accommodate this (unexpected) exigency, the square/diamond symbol containing an inscribed integer is used to associate a multiplication with a *path* rather than a node.

Multiplications by the unit imaginary ($+j$, or $-j$ as appropriate) appear in both the FFT and RFFT diagrams. It has seemed reasonable to count such multiplications separately from those multiplications involving a general complex coefficient, because multiplication of a complex number by j is accomplished by a *complementary swap*, i.e., swap the real and imaginary parts, then negate one of them, as shown in (8).

$$j(a + jb) \equiv -b + ja \quad a, b \text{ real.} \quad (8)$$

Thus, it may be appreciated that multiplication by j is actually a simpler operation than multiplication by a general complex coefficient, and is, in fact, simpler than a complex addition. This is the reason underlying the choice of version of the FFT selected for comparison with the RFFT, since other FFT versions would appear to have *more* complex multiplications, and thus compare less favorably with the RFFT.

The main advantage that might be claimed for the RFFT is reduction of the number of multiplication operations required to implement the transform. That the use of multiplication by real coefficients (as in the RFFT) constitutes a saving over the use of multiplication by complex coefficients is apparent: Since the data to be transformed are assumed complex (worst case for the RFFT), it requires four real multiplications to implement

complex-coefficient multiplication, but only two real multiplications when the coefficient is *a priori* known to be only real-valued. Thus, the 12 real-coefficient multiplications of the RFFT of Fig. 2 represent considerably fewer multiplications than the 10 complex-coefficient multiplications of the FFT of Fig. 1. It is also true that multiplication of a complex number by a complex coefficient requires the equivalent of a complex addition, i.e., $(a + jb)(c + jd) \equiv (ac - bd) + j(bc + ad)$. Thus, even though the RFFT uses 72 additions and the FFT uses 64 additions, it appears that the excess of additions in the RFFT is balanced in the comparison by the fact that 12 additions are implicit in the 12 complex-coefficient multiplications in the FFT.

The FFT of Fig. 1 is of that class of FFTs in which normally ordered data are transformed into a *shuffled-order* transform; i.e., the resulting elements of the transform do not come out in the natural order of their subscripts. (The shuffled-order form of output is one of the accepted canonical forms, cf. Ref. 2.) It will be seen that the RFFT of Fig. 2 also has a shuffled-order output, which has been arranged to show some (incomplete) similarity to the FFT output in Fig. 1. A general rule for the RFFT output arrangement awaits development.

A curious phenomenon of the RFFT which occurs by design is the fact that the coefficients of the conventional discrete Fourier analysis (Ref. 3) in terms of sines and cosines is obtained at intermediate points of the flowgraph: The cosine-harmonic amplitudes are $a_0, a_1, a_2, \dots, a_6, a_7, a_8$ and the sine-harmonic amplitudes are $b_1, b_2, \dots, b_6, b_7$ (b_0 and b_8 are never present), where the subscript indicates the frequency or *harmonic number*. Thus, the RFFT (suitably truncated) provides a "fast" method for obtaining the Fourier analysis.

III. Derivation of the Comparison FFT by Matrix Partitioning

The complex-valued discrete Fourier transform (DFT) is usually defined by the matrix-vector product (9):

$$\begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_{N-1} \end{bmatrix} = \frac{1}{N} \begin{bmatrix} e^{-j 2\pi/N mn} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix} \quad (9)$$

N = dimension of transform matrix
 m = row index ("frequency")
 n = column index ("time")

In (9), the column vector $\{x\}$ on the right is a complex-valued time series of N points (equally spaced in time) which is to be transformed by the indicated product to produce the column vector $\{s\}$, which is called the discrete Fourier transform (DFT) of $\{x\}$. The transform $\{s\}$ is, however, seldom computed by the direct application of the product (9). This is because combinational schemes, called "fast" Fourier transforms (FFT) exist which can compute the DFT with far fewer multiplications and additions than are indicated in (9). One such FFT will be derived in this section.

A. Notation

It is customary in DFT derivations to suppress the factor $1/N$ appearing in (9) and to focus attention on the exponents of the exponential elements of the DFT matrix in (9). Consider the case $N = 16$ to be used in the derivation. The 16×16 DFT can be indicated as shown in Fig. 3a, in which the elements of the matrix (9) are replaced by the exponents mn of the matrix elements which were of the form $[\exp - j(2\pi/16)]^{mn}$, with application of the periodicity identity (1).

The elements of the transform $\{s\}$ can, of course, be computed in any desired order by permuting the order in which the rows of exponents appear in the transform matrix. A convenient arrangement of the rows is that shown in Fig. 3b. This arrangement gives the elements of $\{s\}$ in the so-called "shuffled" order (Ref. 2). The shuffling procedure may be derived by inspection, noting that all rows having 0 in column 8 are sorted into a group at the top of the matrix, so that the remaining rows having 8 in column 8 are sorted to the bottom. The top and bottom halves are then regarded as independent submatrices, and these are further sorted to group together those rows having identical elements in column 4, then column 2. This row sorting operation may be called an "even-odd sort on identical column keys."

B. Derivation

The sorting operation positions the elements of the 16×16 DFT matrix so that the relationships $A_{12} = A_{11}$ and $A_{22} = -A_{21}$ hold among the four 8×8 partition-block submatrices. That $A_{22} = -A_{21}$ follows from the negation identity (2); i.e., if 8 is added to the exponents of A_{21} (modulo 16, of course), the exponents of A_{22} result. Application of the distributive law causes the DFT of Fig. 3b to be reduced to the form shown in Fig. 4. This reduction is accomplished at a cost of 8 additions and 8 subtractions (called 16 "additions"). The reduction leaves two 8×8 matrix multiplications as the indicated operations to be

carried out to complete the transform. But it turns out that these indicated matrix multiplications can themselves be reduced by a procedure essentially the same as that already applied. Thus, the matrix multiplications remaining at each step of reduction are never actually carried out explicitly.

Before continuing to the next step of reduction, it may be noted by inspection of the matrix for the half-transform $(s_1, s_9, \dots, s_7, s_{15})$ that if exponent 4 is factored from the last four columns, then relationships similar to the $A_{12} = A_{11}$ and $A_{22} = -A_{21}$ of the preceding step hold among the 4×4 partition-blocks. The factoring is most efficiently accomplished by multiplying the last four elements of the column vector by $\{\exp[-j(2\pi/16)]\}^4 = -j$, as indicated schematically by the circle symbol $\textcircled{4}$, and subtracting 4 from the elements of the last four matrix columns. It may now be seen that column vectors on the right in Fig. 4 are precisely those available at the first echelon of nodes in the FFT flowchart of Fig. 1. The principle of factoring entire columns of matrix elements by means of multiplication of the corresponding vector elements is the key also to the RFFT derivation given in the next section. In the present example, the fact that exponents of 0 and 8 remain in column four after the factorization causes no difficulty, since exponent 0 is *unity* and exponent 8 is *negation*, as used in the first reduction step. To avoid carrying unduly complicated notation to the next reduction step, the vectors on which matrix operations are indicated are redesignated as $\{y\}$, is shown in Fig. 4.

Figure 5 gives the remaining reduction steps. The procedure is the same as for the first reduction step. At each step, the notations for vectors representing intermediate computations are redefined to simplify the notation for the next step. The result is shown in the FFT flowchart of Fig. 1. It is seen that N ($N = 16$ in the example) additions are required at each step of the reduction, and that exactly $N \log_2 N$ additions (if N is a power of 2) are required. (Here, $N \log_2 N = 64$). The number of multiplications by a complex scalar coefficient is 10 if multiplications by $-j$ (unit imaginary) are not counted (as indicated in the preceding section); otherwise, the total number of multiplications is 17.

C. Theorem

The count C of nontrivial multiplications by complex coefficients in an FFT flowchart based on the described DFT reduction scheme for N a power of 2 is

$$C = \frac{N}{2} \log_2 \frac{N}{8} + 2 \quad 2 \leq N$$

The proof is by inspection. Temporarily assume $8 \leq N$. The number of echelons, or reduction steps is $\log_2 N$. The first echelon requires no nontrivial multiplications, and the last echelon requires no multiplications at all. Therefore, only $E = (\log_2 N - 2) = \log_2 (N/4)$ intermediate echelons require multiplication.

There are always less than $N/2$ multiplications in each echelon: Inspecting the E intermediate echelons, it may be seen that the lower half of the first (intermediate) echelon requires multiplication at half its nodes. Then, the lower $3/4$ of the next echelon requires multiplication at half its nodes. Then the lower $7/8$ of the next echelon requires multiplication at half its nodes, etc. Since the width of the echelon is N , the number of multiplications in the flowchart is, assuming the induction,

$$C = \frac{1}{2} N \left(\frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \dots \right)_{E \text{ terms}}$$

Complementing the indicated summation gives

$$\begin{aligned} C &= \frac{1}{2} N \left[E - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \right)_{E \text{ terms}} \right] \\ &= \frac{1}{2} N \left[E - \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{4} + \dots \right)_{E \text{ terms}} \right] \end{aligned}$$

Summing the indicated geometric series to E terms yields

$$C = \frac{1}{2} N \left[E - \frac{1}{2} \left(\frac{(1/2)^E - 1}{(1/2) - 1} \right) \right] = \frac{1}{2} N [E - 1 + (1/2)^E]$$

Substituting $E = \log_2 (N/4) \geq 1$, the theorem is proved for $8 \leq N$. Construction of the flowgraph for the simple cases $N = 2$ and $N = 4$ establishes that the formula is valid for $2 \leq N$.

IV. Derivation of the RFFT

In this section, the RFFT is derived from the DFT definition (9), with factor $1/N$ suppressed. This is done by reducing the matrix in (9) to real terms to a maximum extent, introducing explicit factors of j where required, and then reducing the indicated real matrix operations by application of the distributive law and scalar factorization of matrix columns, similarly to what was done in the preceding section.

A. Notation

The matrix (9) for the DFT is reduced to real terms as shown schematically in Fig. 6. The matrix (9) is ex-

pressed directly in Fig. 6a. Through application of the aliasing identity (4), the positive-frequency exponentials in the frequency range $(N/2) + 1$ to $N - 1$ are expressed in terms of negative-frequency exponentials in the frequency range $-[(N/2) - 1]$ to -1 , as shown in Fig. 6b, for general N (even). (This modification has been applied to the lower $(N/2) - 1$ rows). Without loss of generality, application of the definition (5) to the matrix of Fig. 6b produces, for $N = 16$, the matrix of Fig. 6c. It may be observed that there are no sine terms at frequencies 0 and 8. Also, the elements of the row for $\cos[(2\pi/16)0n]$ are all equal to unity (real) and the elements of the row for $\cos[(2\pi/16)8n]$ are alternating $+1, -1, +1, -1, \dots$ (real).

It appears then that the complex DFT can be written in terms of a real sine-cosine matrix, with the aid of a simple factorization, as shown in Fig. 7a. The product of the sine-cosine matrix times the data vector $\{x\}$ can itself be regarded as a transform. This transform will be called the *discrete Fourier analysis* (DFA), and is seen to be a matrix formalism for conventional discrete Fourier analysis (Ref. 3). The coefficients $a_0, a_1, a_2, \dots, a_6, a_7, a_8$ and $b_1, b_2, \dots, b_6, b_7$ are, respectively, the discrete Fourier cosine and sine amplitude coefficients. These coefficients are real if the data vector $\{x\}$ is real, and complex if $\{x\}$ is complex. The DFA matrix elements are, of course, strictly real-valued.

The notation to be used may now be introduced: Each element of the DFA matrix is a value of a sine or cosine function at some value of argument. These values will be indicated by a simple *code* which will express every value in the matrix in terms of the sine function tabulated for the first quadrant of its argument. The code is, for $N = 16$,

Code	Value
numerals $\left\{ \begin{array}{l} 0 \\ 1 \\ 2 \\ 3 \end{array} \right.$	$\sin\left(\frac{2\pi}{16} 0\right) = 0$ $\sin\left(\frac{2\pi}{16} 1\right)$ $\sin\left(\frac{2\pi}{16} 2\right)$ $\sin\left(\frac{2\pi}{16} 3\right)$
letter I	$\sin\left(\frac{2\pi}{16} 4\right) = 1$

It will be necessary to show negation of the encoded sine values, and this can conveniently be done by under-

lining. This notation may then be applied to the DFA matrix of Fig. 7b, with the aid of identities (6) and (7), to obtain the DFA matrix form shown in Fig. 8a.

B. Derivation

The RFFT is defined as the "fast" computational organization of the operations indicated in matrix form in Fig. 7a. Since the transformation matrix on the left in Fig. 7a can be accomplished with but 7 trivial multiplications by j (or $-j$) and 14 additions, this transformation is already in "fast" form. Thus, it will be necessary only to derive a "fast" version of the DFA of Fig. 7b. This will be done using the notation of Fig. 8a.

As was done in the derivation of the FFT in Section III, the outputs of the DFA in Fig. 8a will be *shuffled* by permuting the rows of the DFA matrix to the form shown in Fig. 8b. The shuffling is done to enhance symmetries among the 8×8 partition blocks. Application of the distributive law to the partition blocks of the matrix of Fig. 8b yields the reduced transform of Fig. 9a. It may be observed that the sums and differences of x_i appearing on the right hand side of Fig. 9a are precisely the quantities available at the first echelon of nodes in the RFFT flowchart of Fig. 2. These sums and differences are redesignated as $\{y\}$ to simplify the notation for the second reduction step, shown in Fig. 9b.

The reduction from the lower 8×8 matrix of Fig. 9a to the two lower 4×4 matrices of Fig. 9b departs somewhat from the straightforward application of the distributive law, as do the subsequent reduction steps, in that *folding* about the pivotal elements y_8 and y_{12} is required. What is meant by folding can be inferred from the lower two 4×4 matrix operations in Fig. 9b, which recognize the odd symmetry which holds among the columns of rows 0 through 3 and the even symmetry which holds among the columns of rows 4 through 7, respectively, in the lower 8×8 matrix of Fig. 9a. Before proceeding to the third reduction step, the value $\sin[(2\pi/16) \cdot 2]$ is factored from the third column of the two lower 4×4 matrices in Fig. 9b, and is applied as indicated by the ② symbol to the corresponding column-vector elements. The column vector notation is then changed to $\{z\}$ to simplify the notation for subsequent steps.

In the third reduction step, shown in Fig. 10, the distributive law is applied directly to the matrix operations for a_0, a_8, a_4 , and b_4 . The matrix rows for a_2, a_6 are folded to exploit odd column symmetry, and the rows for b_2, b_6 are folded to exploit even column symmetry with

column factorization applied. The matrix operations for the pairs (a_1, a_7) , (a_5, a_3) , (b_1, b_7) , and (b_5, b_3) are reduced, as shown in Fig. 10, by matrix partitioning and column factorization.

The fourth (final) reduction step can be done by inspection, and so is not shown in matrix form. All of the reduction steps are, of course, equivalent to the DFA portion of the RFFT flowgraph of Fig. 2, to which has been added the transformation which converts the DFA to the RFFT.

V. Concluding Remarks

The code notation for the sine function, used in the RFFT derivation, has turned out to be similar to that used in Ref. 4. A distinction, however, is in the fact that the sine code used in the present report is *value-oriented*,

and has been applied to give the cosine as well as the sine, tabular values. Indeed, the results of row shuffling, as in Fig. 8a, show that the sine and cosine amplitude coefficients are most efficiently obtained when interspersed to some extent, and the value notation gives the *sorting keys* for doing this.

An unelaborated comment in Ref. 5 suggested that the complex exponentials of the FFT matrix be expressed in real form, to minimize the number of real multiplications. The RFFT derivation realizes such a minimization. Also suggested in Ref. 5 was the possibility of regarding the unit imaginary j as a "special element" to be exploited in FFT matrix factorization schemes, and this has been done to a maximum extent in the derivation of the comparison FFT. The importance of j is not, of course, its use as a pivotal element of the factorization, but rests rather with the fact that multiplication by j is a trivial complex multiplication, to be exploited in the actual computations.

References

1. Cooley, J. W., and Tukey, J. W., "An algorithm for the machine calculation of complex Fourier series," *Math. Comput.*, Vol. 19, pp 297-301, Apr. 1965.
2. Cochran, W. T., et al., "What is the fast Fourier transform?," *IEEE Trans. Audio and Electroacoust.*, Vol. AU-15, No. 2, pp. 45-55, June 1967.
3. Korn, G. A., and Korn, T. M., *Mathematical Handbook for Scientists and Engineers*, McGraw-Hill Book Company, New York, 1961.
4. Danielson, G. C., and Lanczos, C., "Some improvements in practical Fourier analysis and their application to X-ray scattering from liquids," *J. Franklin Inst.*, Vol. 233, pp. 365-380, Apr. 1942.
5. Gentleman, W. M., "Matrix multiplication and fast Fourier transforms," *Bell Sys. Tech. J.*, Vol. 47, pp. 1099-1103, July 1968.

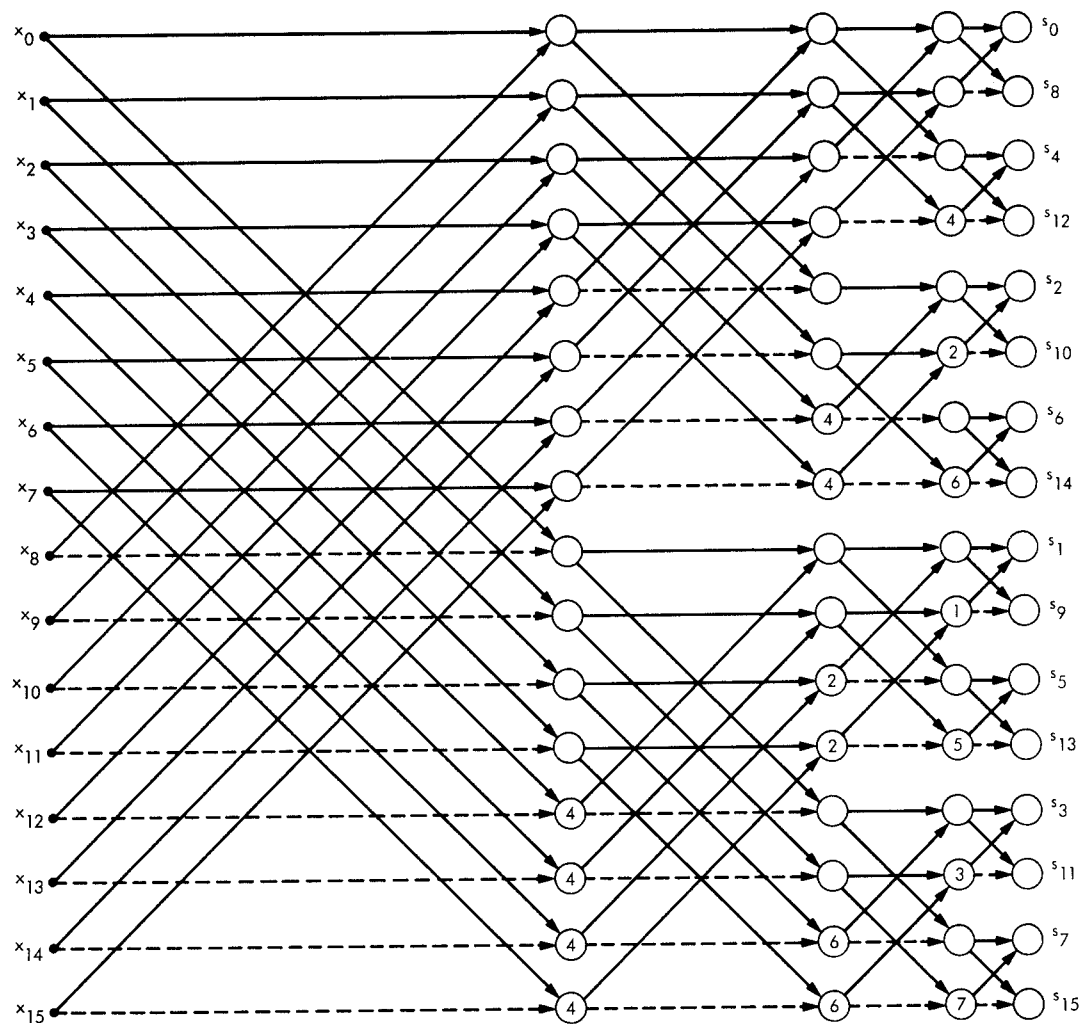


Fig. 1. Flowchart for conventional Fast Fourier Transform ($N = 16$)

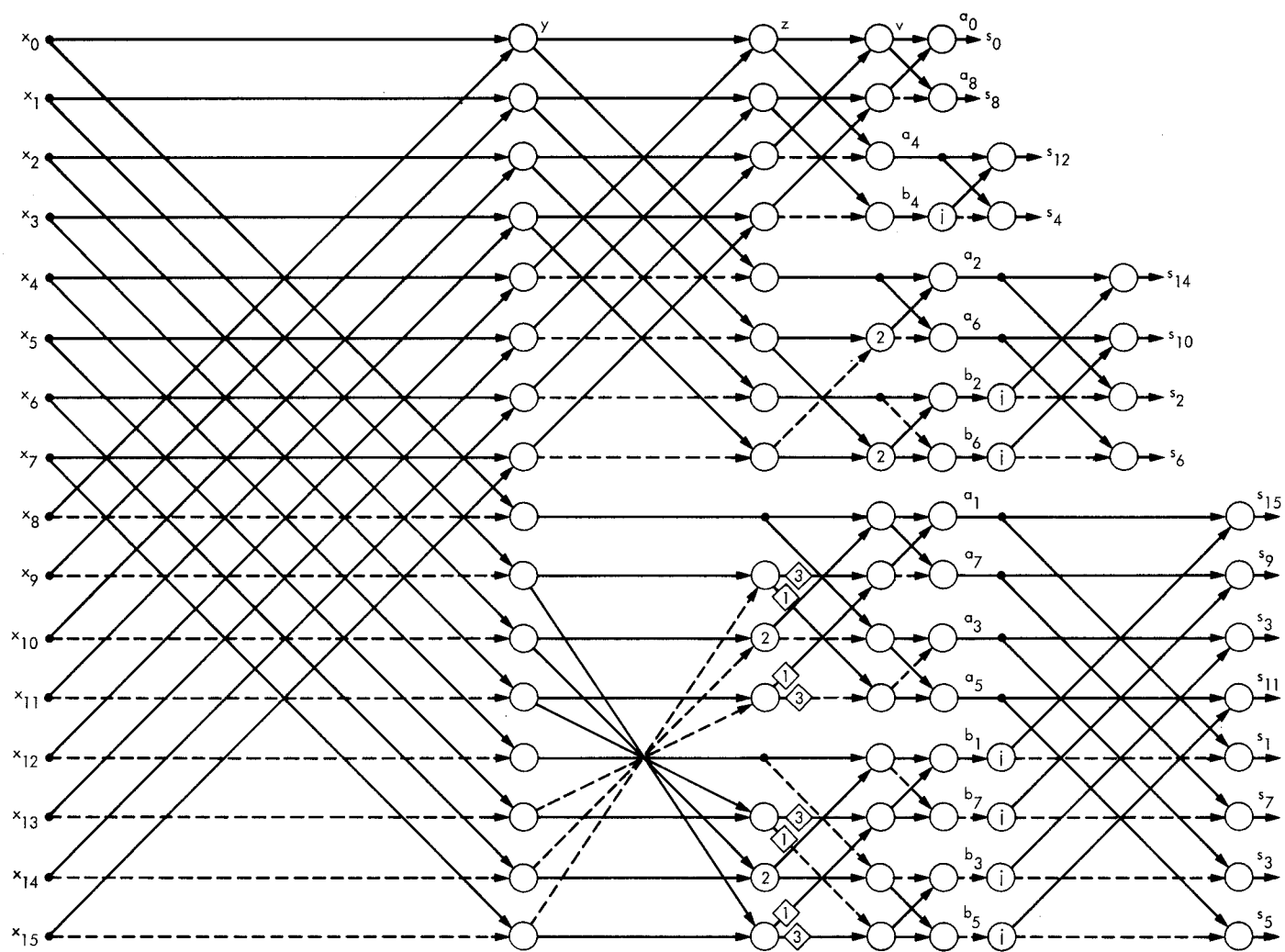


Fig. 2. Flowchart for RFFT

$$\begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \\ s_8 \\ s_9 \\ s_{10} \\ s_{11} \\ s_{12} \\ s_{13} \\ s_{14} \\ s_{15} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 \\ 0 & 3 & 6 & 9 & 12 & 15 & 2 & 5 & 8 & 11 & 14 & 1 & 4 & 7 & 10 & 13 \\ 0 & 4 & 8 & 12 & 0 & 4 & 8 & 12 & 0 & 4 & 8 & 12 & 0 & 4 & 8 & 12 \\ 0 & 5 & 10 & 15 & 4 & 9 & 14 & 3 & 8 & 13 & 2 & 7 & 12 & 1 & 6 & 11 \\ 0 & 6 & 12 & 2 & 8 & 14 & 4 & 10 & 0 & 6 & 12 & 2 & 8 & 14 & 4 & 10 \\ 0 & 7 & 14 & 5 & 12 & 3 & 10 & 1 & 8 & 15 & 6 & 13 & 4 & 11 & 2 & 9 \\ 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8 \\ 0 & 9 & 2 & 11 & 4 & 13 & 6 & 15 & 8 & 1 & 10 & 3 & 12 & 5 & 14 & 7 \\ 0 & 10 & 4 & 14 & 8 & 2 & 12 & 6 & 0 & 10 & 4 & 14 & 8 & 2 & 12 & 6 \\ 0 & 11 & 6 & 1 & 12 & 7 & 2 & 13 & 8 & 3 & 14 & 9 & 4 & 15 & 10 & 5 \\ 0 & 12 & 8 & 4 & 0 & 12 & 8 & 4 & 0 & 12 & 8 & 4 & 0 & 12 & 8 & 4 \\ 0 & 13 & 10 & 7 & 4 & 1 & 14 & 11 & 8 & 5 & 2 & 15 & 12 & 9 & 6 & 3 \\ 0 & 14 & 12 & 10 & 8 & 6 & 4 & 2 & 0 & 14 & 12 & 10 & 8 & 6 & 4 & 2 \\ 0 & 15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{bmatrix}$$

(a) NORMAL-ORDER DFT FOR N = 16

$$\begin{bmatrix} s_0 \\ s_8 \\ s_4 \\ s_{12} \\ s_2 \\ s_{10} \\ s_6 \\ s_{14} \\ s_1 \\ s_9 \\ s_5 \\ s_{13} \\ s_3 \\ s_{11} \\ s_7 \\ s_{15} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8 \\ 0 & 4 & 8 & 12 & 0 & 4 & 8 & 12 & 0 & 4 & 8 & 12 & 0 & 4 & 8 & 12 \\ 0 & 12 & 8 & 4 & 0 & 12 & 8 & 4 & 0 & 12 & 8 & 4 & 0 & 12 & 8 & 4 \\ 0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 \\ 0 & 10 & 4 & 14 & 8 & 2 & 12 & 6 & 0 & 10 & 4 & 14 & 8 & 2 & 12 & 6 \\ 0 & 6 & 12 & 2 & 8 & 14 & 4 & 10 & 0 & 6 & 12 & 2 & 8 & 14 & 4 & 10 \\ 0 & 14 & 12 & 10 & 8 & 6 & 4 & 2 & 0 & 14 & 12 & 10 & 8 & 6 & 4 & 2 \\ \hline 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 0 & 9 & 2 & 11 & 4 & 13 & 6 & 15 & 8 & 1 & 10 & 3 & 12 & 5 & 14 & 7 \\ 0 & 5 & 10 & 15 & 4 & 9 & 14 & 3 & 8 & 13 & 2 & 7 & 12 & 1 & 6 & 11 \\ 0 & 13 & 10 & 7 & 4 & 1 & 14 & 11 & 8 & 5 & 2 & 15 & 12 & 9 & 6 & 3 \\ 0 & 3 & 6 & 9 & 12 & 15 & 2 & 5 & 8 & 11 & 14 & 1 & 4 & 7 & 10 & 13 \\ 0 & 11 & 6 & 1 & 12 & 7 & 2 & 13 & 8 & 3 & 14 & 9 & 4 & 15 & 10 & 5 \\ 0 & 7 & 14 & 5 & 12 & 3 & 10 & 1 & 8 & 15 & 6 & 13 & 4 & 11 & 2 & 9 \\ 0 & 15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{bmatrix}$$

(b) SHUFFLED-ORDER DFT FOR N = 16, OBTAINED BY PERMUTING THE ROWS OF THE MATRIX IN (a).

Fig. 3. Discrete Fourier transforms for N = 16 using integer exponents as symbolic representations of matrix elements

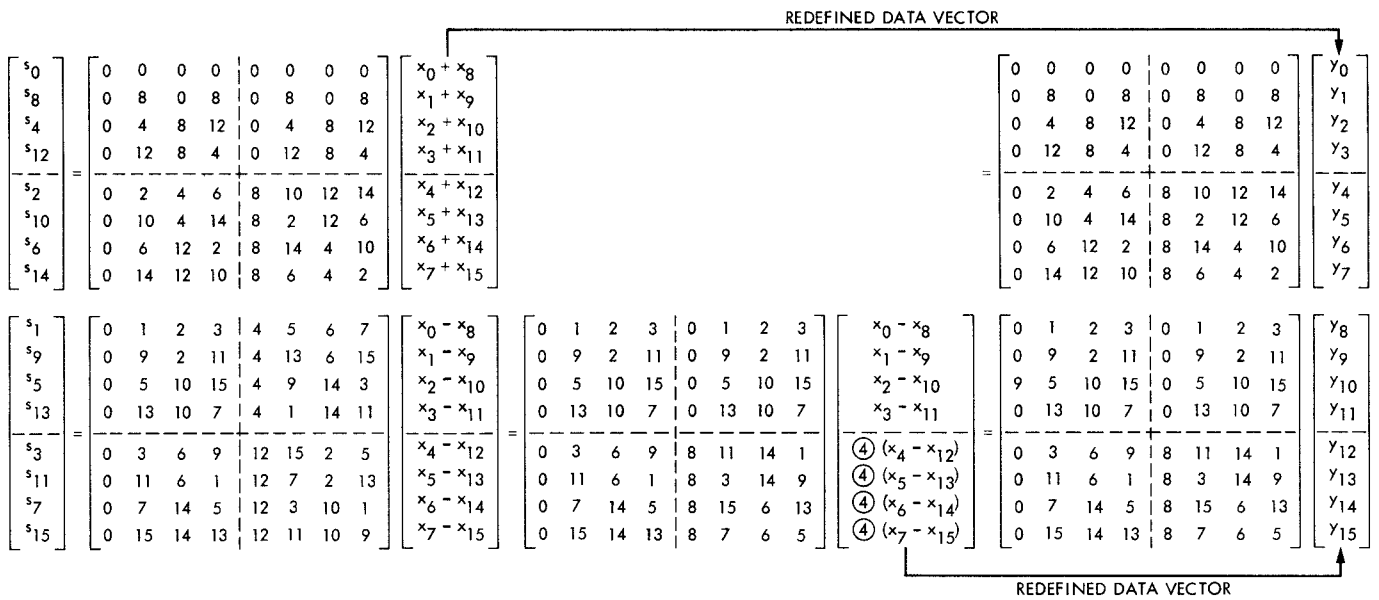


Fig. 4. First step of reduction of DFT of Fig. 3

REDEFINED DATA VECTORS

$$\begin{aligned}
 \begin{bmatrix} s_0 \\ s_8 \\ s_4 \\ s_{12} \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 8 \\ 0 & 4 & 8 & 12 \\ 0 & 12 & 8 & 4 \end{bmatrix} \begin{bmatrix} y_0 + y_4 \\ y_1 + y_5 \\ y_2 + y_6 \\ y_3 + y_7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 8 \\ 0 & 4 & 8 & 12 \\ 0 & 12 & 8 & 4 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} \\
 \begin{bmatrix} s_2 \\ s_{10} \\ s_6 \\ s_{14} \end{bmatrix} &= \begin{bmatrix} 0 & 2 & 4 & 6 \\ 0 & 10 & 4 & 14 \\ 0 & 6 & 12 & 2 \\ 0 & 14 & 12 & 10 \end{bmatrix} \begin{bmatrix} y_0 - y_4 \\ y_1 - y_5 \\ y_2 - y_6 \\ y_3 - y_7 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 2 \\ 0 & 10 & 0 & 10 \\ 0 & 6 & 8 & 14 \\ 0 & 14 & 8 & 6 \end{bmatrix} \begin{bmatrix} y_0 - y_4 \\ y_1 - y_5 \\ \textcircled{4}(y_2 - y_6) \\ \textcircled{4}(y_3 - y_7) \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 2 \\ 0 & 10 & 0 & 10 \\ 0 & 6 & 8 & 14 \\ 0 & 14 & 8 & 6 \end{bmatrix} \begin{bmatrix} z_4 \\ z_5 \\ z_6 \\ z_7 \end{bmatrix} \\
 \begin{bmatrix} s_1 \\ s_9 \\ s_5 \\ s_{13} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 9 & 2 & 11 \\ 0 & 5 & 10 & 15 \\ 0 & 13 & 10 & 7 \end{bmatrix} \begin{bmatrix} y_8 + y_{12} \\ y_9 + y_{13} \\ y_{10} + y_{14} \\ y_{11} + y_{15} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 9 & 0 & 9 \\ 0 & 5 & 8 & 13 \\ 0 & 13 & 8 & 5 \end{bmatrix} \begin{bmatrix} y_8 + y_{12} \\ y_9 + y_{13} \\ \textcircled{2}(y_{10} + y_{14}) \\ \textcircled{2}(y_{11} + y_{15}) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 9 & 0 & 9 \\ 0 & 5 & 8 & 13 \\ 0 & 13 & 8 & 5 \end{bmatrix} \begin{bmatrix} z_8 \\ z_9 \\ z_{10} \\ z_{11} \end{bmatrix} \\
 \begin{bmatrix} s_3 \\ s_{11} \\ s_7 \\ s_{15} \end{bmatrix} &= \begin{bmatrix} 0 & 3 & 6 & 9 \\ 0 & 11 & 6 & 1 \\ 0 & 7 & 14 & 5 \\ 0 & 15 & 14 & 13 \end{bmatrix} \begin{bmatrix} y_8 - y_{12} \\ y_9 - y_{13} \\ y_{10} - y_{14} \\ y_{11} - y_{15} \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 & 3 \\ 0 & 11 & 0 & 11 \\ 0 & 7 & 8 & 15 \\ 0 & 15 & 8 & 7 \end{bmatrix} \begin{bmatrix} y_8 - y_{12} \\ y_9 - y_{13} \\ \textcircled{6}(y_{10} - y_{14}) \\ \textcircled{6}(y_{11} - y_{15}) \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 & 3 \\ 0 & 11 & 0 & 11 \\ 0 & 7 & 8 & 15 \\ 0 & 15 & 8 & 7 \end{bmatrix} \begin{bmatrix} z_{12} \\ z_{13} \\ z_{14} \\ z_{15} \end{bmatrix}
 \end{aligned}$$

(a) REDEFINED DATA VECTORS

$$\begin{aligned}
 \begin{bmatrix} s_0 \\ s_8 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} z_0 + z_2 \\ z_1 + z_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \\
 \begin{bmatrix} s_4 \\ s_{12} \end{bmatrix} &= \begin{bmatrix} 0 & 4 \\ 0 & 12 \end{bmatrix} \begin{bmatrix} z_1 - z_2 \\ z_1 - z_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} z_0 - z_2 \\ \textcircled{4}(z_1 - z_3) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} v_2 \\ v_3 \end{bmatrix} \\
 \begin{bmatrix} s_2 \\ s_{10} \end{bmatrix} &= \begin{bmatrix} 0 & 2 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} z_4 + z_6 \\ z_5 + z_7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} z_4 + z_6 \\ \textcircled{2}(z_5 + z_7) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} v_4 \\ v_5 \end{bmatrix} \\
 \begin{bmatrix} s_6 \\ s_{14} \end{bmatrix} &= \begin{bmatrix} 0 & 6 \\ 0 & 14 \end{bmatrix} \begin{bmatrix} z_4 - z_6 \\ z_5 - z_7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} z_4 - z_6 \\ \textcircled{6}(z_5 - z_7) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} v_6 \\ v_7 \end{bmatrix} \\
 \begin{bmatrix} s_1 \\ s_9 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} z_8 + z_{10} \\ z_9 + z_{11} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} z_8 + z_{10} \\ \textcircled{1}(z_9 + z_{11}) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} v_8 \\ v_9 \end{bmatrix} \\
 \begin{bmatrix} s_5 \\ s_{13} \end{bmatrix} &= \begin{bmatrix} 0 & 5 \\ 0 & 13 \end{bmatrix} \begin{bmatrix} z_8 - z_{10} \\ z_9 - z_{11} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} z_8 - z_{10} \\ \textcircled{5}(z_9 - z_{11}) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} v_{10} \\ v_{11} \end{bmatrix} \\
 \begin{bmatrix} s_3 \\ s_{11} \end{bmatrix} &= \begin{bmatrix} 0 & 3 \\ 0 & 11 \end{bmatrix} \begin{bmatrix} z_{12} + z_{14} \\ z_{13} + z_{15} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} z_{12} + z_{14} \\ \textcircled{3}(z_{13} + z_{15}) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} v_{12} \\ v_{13} \end{bmatrix} \\
 \begin{bmatrix} s_7 \\ s_{15} \end{bmatrix} &= \begin{bmatrix} 0 & 7 \\ 0 & 15 \end{bmatrix} \begin{bmatrix} z_{12} - z_{14} \\ z_{13} - z_{15} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} z_{12} - z_{14} \\ \textcircled{7}(z_{13} - z_{15}) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} v_{14} \\ v_{15} \end{bmatrix}
 \end{aligned}$$

(b) FINAL STEP (4)

$$\begin{aligned}
 \begin{bmatrix} s_0 \\ s_8 \\ s_4 \\ s_{12} \\ s_2 \\ s_{10} \\ s_6 \\ s_{14} \\ s_1 \\ s_9 \\ s_5 \\ s_{13} \\ s_3 \\ s_{11} \\ s_7 \\ s_{15} \end{bmatrix} &= \begin{bmatrix} v_0 + v_1 \\ v_0 - v_1 \\ v_2 + v_3 \\ v_2 - v_3 \\ v_4 + v_5 \\ v_4 - v_5 \\ v_6 + v_7 \\ v_6 - v_7 \\ v_8 + v_9 \\ v_8 - v_9 \\ v_{10} + v_{11} \\ v_{10} - v_{11} \\ v_{12} + v_{13} \\ v_{12} - v_{13} \\ v_{14} + v_{15} \\ v_{14} - v_{15} \end{bmatrix}
 \end{aligned}$$

(c)

Fig. 5. Steps 2, 3, and 4 of reduction of DFT

$$\begin{bmatrix}
 \exp(-j 2\pi/N 0 n) \\
 \exp(-j 2\pi/N 1 n) \\
 \vdots \\
 \exp(-j 2\pi/N m n) \\
 \vdots \\
 \exp[-j 2\pi/N (N-1) n]
 \end{bmatrix}$$

NOW INDEX m IS FREQUENCY $0 \leq m \leq N-1$
 COLUMN INDEX n IS TIME $0 \leq n \leq N-1$

(a)

$$\begin{bmatrix}
 \exp(-j 2\pi/N 0 n) \\
 \exp(-j 2\pi/N 1 n) \\
 \exp(-j 2\pi/N 2 n) \\
 \vdots \\
 \exp(+j 2\pi/N \cdot N/2) \\
 \exp[+j 2\pi/N (N/2 - 1)] \\
 \vdots \\
 \exp(+j 2\pi/N 3 n) \\
 \exp(+j 2\pi/N 2 n) \\
 \exp(+j 2\pi/N 1 n)
 \end{bmatrix}$$

(b)

LEGEND FOR (a), (b):
 m IS ROW INDEX AND POSITIVE FREQUENCY $0 \leq m \leq N-1$
 n IS COLUMN INDEX AND TIME $0 \leq n \leq N-1$
 N IS DIMENSION OF MATRIX

$$\begin{bmatrix}
 \cos(2\pi/16 0 n) \\
 \cos(2\pi/16 1 n) - j \sin(2\pi/16 1 n) \\
 \cos(2\pi/16 2 n) - j \sin(2\pi/16 2 n) \\
 \vdots \\
 \cos(2\pi/16 7 n) - j \sin(2\pi/16 7 n) \\
 \cos(2\pi/16 8 n) \\
 \cos(2\pi/16 7 n) + j \sin(2\pi/16 7 n) \\
 \vdots \\
 \cos(2\pi/16 2 n) + j \sin(2\pi/16 2 n) \\
 \cos(2\pi/16 1 n) + j \sin(2\pi/16 1 n)
 \end{bmatrix}$$

(c)

CASE $N = 16$

Fig. 6. Reduction of DFT matrix to sine-cosine form

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 1 & 3 & 2 & 1 & 0 & 1 & 2 & 3 \\ 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 \\ 1 & 1 & 2 & 3 & 0 & 3 & 2 & 1 & 1 & 1 & 2 & 3 & 0 & 3 & 2 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 3 & 0 & 3 & 2 & 1 & 1 & 1 & 2 & 3 & 0 & 3 & 2 & 1 \\ 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 \\ 1 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 1 & 3 & 2 & 1 & 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 1 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 1 & 3 & 2 & 1 \\ 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 \\ 0 & 3 & 2 & 1 & 1 & 1 & 2 & 3 & 0 & 3 & 2 & 1 & 1 & 1 & 2 & 3 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 3 & 2 & 1 & 1 & 1 & 2 & 3 & 0 & 3 & 2 & 1 & 1 & 1 & 2 & 3 \\ 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 \\ 0 & 1 & 2 & 3 & 1 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 1 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{bmatrix}$$

(a) NORMAL-ORDER DFA

$$\begin{bmatrix} a_0 \\ a_8 \\ a_4 \\ b_4 \\ a_2 \\ a_6 \\ b_2 \\ b_6 \\ a_1 \\ a_7 \\ a_5 \\ a_3 \\ b_1 \\ b_7 \\ b_5 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 \\ 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 \\ 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 \\ 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 \\ 1 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 1 & 3 & 2 & 1 & 0 & 1 & 2 & 3 \\ 1 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 1 & 3 & 2 & 1 & 0 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 & 0 & 3 & 2 & 1 & 1 & 1 & 2 & 3 & 0 & 3 & 2 & 1 \\ 1 & 1 & 2 & 3 & 0 & 3 & 2 & 1 & 1 & 1 & 2 & 3 & 0 & 3 & 2 & 1 \\ 0 & 1 & 2 & 3 & 1 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 1 & 3 & 2 & 1 \\ 0 & 1 & 2 & 3 & 1 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 1 & 3 & 2 & 1 \\ 0 & 3 & 2 & 1 & 1 & 1 & 2 & 3 & 0 & 3 & 2 & 1 & 1 & 1 & 2 & 3 \\ 0 & 3 & 2 & 1 & 1 & 1 & 2 & 3 & 0 & 3 & 2 & 1 & 1 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{bmatrix}$$

(b) SHUFFLED-ORDER DFA

Fig. 8. Discrete Fourier analysis (DFA) for $N = 16$ using integer code for quadrantal sine tabular values

$$\begin{bmatrix} a_0 \\ a_8 \\ a_4 \\ b_4 \\ a_2 \\ a_6 \\ b_2 \\ b_6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 \\ 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 \\ 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 \\ 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_0 + x_8 \\ x_1 + x_9 \\ x_2 + x_{10} \\ x_3 + x_{11} \\ x_4 + x_{12} \\ x_5 + x_{13} \\ x_6 + x_{14} \\ x_7 + x_{15} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_7 \\ a_5 \\ a_3 \\ b_1 \\ b_7 \\ b_5 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 1 & 2 & 3 \\ 1 & 3 & 2 & 1 & 0 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 & 0 & 3 & 2 & 1 \\ 1 & 1 & 2 & 3 & 0 & 3 & 2 & 1 \\ 0 & 1 & 2 & 3 & 1 & 3 & 2 & 1 \\ 0 & 1 & 2 & 3 & 1 & 3 & 2 & 1 \\ 0 & 3 & 2 & 1 & 1 & 1 & 2 & 3 \\ 0 & 3 & 2 & 1 & 1 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_0 - x_8 \\ x_1 - x_9 \\ x_2 - x_{10} \\ x_3 - x_{11} \\ x_4 - x_{12} \\ x_5 - x_{13} \\ x_6 - x_{14} \\ x_7 - x_{15} \end{bmatrix} = \begin{bmatrix} y_8 \\ y_9 \\ y_{10} \\ y_{11} \\ y_{12} \\ y_{13} \\ y_{14} \\ y_{15} \end{bmatrix}$$

(a) REDEFINED INTERMEDIATE VARIABLES

$$\begin{bmatrix} a_0 \\ a_8 \\ a_4 \\ b_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_0 + y_4 \\ y_1 + y_5 \\ y_2 + y_6 \\ y_3 + y_7 \end{bmatrix} = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$\begin{bmatrix} a_2 \\ a_6 \\ b_2 \\ b_6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 1 & 2 & 0 & 2 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} y_0 - y_4 \\ y_1 - y_5 \\ y_2 - y_6 \\ y_3 - y_7 \end{bmatrix} = \begin{bmatrix} z_4 \\ z_5 \\ z_6 \\ z_7 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_7 \\ a_5 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 1 & 3 & 2 & 1 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} y_8 \\ y_9 - y_{15} \\ y_{10} - y_{14} \\ y_{11} - y_{13} \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} y_8 \\ y_9 - y_{15} \\ \textcircled{2}(y_{10} - y_{14}) \\ y_{11} - y_{13} \end{bmatrix} = \begin{bmatrix} z_8 \\ z_9 \\ z_{10} \\ z_{11} \end{bmatrix}$$

$$\begin{bmatrix} b_1 \\ b_7 \\ b_5 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 1 & 3 & 2 & 1 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} y_{12} \\ y_{11} + y_{13} \\ y_{10} + y_{14} \\ y_9 + y_{15} \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} y_{12} \\ y_{11} + y_{13} \\ \textcircled{2}(y_{10} + y_{14}) \\ y_9 + y_{15} \end{bmatrix} = \begin{bmatrix} z_{12} \\ z_{13} \\ z_{14} \\ z_{15} \end{bmatrix}$$

(b)

Fig. 9. First and second reduction steps of the DFA for $N = 16$

$$\begin{array}{l}
 \begin{bmatrix} a_0 \\ a_8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z_0 + z_2 \\ z_1 + z_3 \end{bmatrix} \\
 \begin{bmatrix} a_4 \\ b_4 \end{bmatrix} = \begin{bmatrix} z_0 - z_2 \\ z_1 - z_3 \end{bmatrix} \\
 \begin{bmatrix} a_2 \\ a_6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} z_4 \\ z_5 - z_7 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z_4 \\ \textcircled{2} (z_5 - z_7) \end{bmatrix} \\
 \begin{bmatrix} b_2 \\ b_6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} z_6 \\ z_5 + z_7 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z_6 \\ \textcircled{2} (z_5 + z_7) \end{bmatrix} \\
 \begin{bmatrix} a_1 \\ a_7 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 1 & 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_8 \\ z_9 \\ z_{10} \\ z_{11} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_8 \\ \textcircled{3} z_9 \\ \frac{z_{10}}{\textcircled{1} z_{11}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z_8 + z_{10} \\ \textcircled{3} z_9 + \textcircled{1} z_{11} \end{bmatrix} \\
 \begin{bmatrix} a_5 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} z_8 \\ z_9 \\ z_{10} \\ z_{11} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_8 \\ \textcircled{1} z_9 \\ \frac{z_{10}}{\textcircled{3} z_{11}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z_8 - z_{10} \\ \textcircled{1} z_9 - \textcircled{3} z_{11} \end{bmatrix} \\
 \begin{bmatrix} b_1 \\ b_7 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 1 & 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_{12} \\ z_{13} \\ z_{14} \\ z_{15} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_{12} \\ \textcircled{3} z_{13} \\ \frac{z_{14}}{\textcircled{1} z_{15}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z_{14} + z_{12} \\ \textcircled{1} z_{15} + \textcircled{3} z_{13} \end{bmatrix} \\
 \begin{bmatrix} b_5 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} z_{12} \\ z_{13} \\ z_{14} \\ z_{15} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_{12} \\ \textcircled{1} z_{13} \\ \frac{z_{14}}{\textcircled{3} z_{15}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z_{14} - z_{12} \\ \textcircled{3} z_{15} - \textcircled{1} z_{13} \end{bmatrix}
 \end{array}
 \begin{array}{l}
 \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \\ v_{15} \end{bmatrix}
 \end{array}$$

Fig. 10. Third reduction step of the DFA for $N = 16$